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ON CERTAIN PROPERTIES OF A SUBSONIC FLOW BEHIND A SHOCK WAVE ARISING DURING SUPERSONIC FLOW AROUND BODIES OF FINITE THICKNESS†

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The subsonic vortex flow behind the detached or attached shock wave (SW) which arises during the supersonic planar flow around symmetric bodies of finite thickness is considered. As in [1, 2], the fundamental proofs rest on an analysis of the lines of constant pressure (isobars) in the subsonic domain between the body, the SW and the sonic line joining the SW to the body.

THE IMPOSSIBILITY of certain schemes of planar vortex flow, in particular, with local supersonic zones is proved in Sec. 1.

The mutual relationship between the subsonic segments of the body and the SW is investigated in Sec. 2. It is proved that, in the case of non-negative angles of inclination of the wall, the angles of inclination of the velocity vector behind the SW are also non-negative, the angles of inclination of the SW do not exceed $\pi/2$ and, in the case of a detached shock wave, the pressure increases monotonically along the segment of the axis of symmetry between the SW and the body. If the wall of the body is nonconcave or the angles of inclination of the wall exceed the limiting angle for the shock polar, the subsonic segment of the SW is concave and, here, the pressure decreases monotonically along it.

The circumfluence of a finite wedge with a salient point of the generatrix is considered in Sec. 3. It is proved that the sonic line (SL) joining the body to the shock wave emerges from the salient point of the generatrix and also that the pressure decreases monotonically along the wall.

1. Let us consider the planar vortex flow of an ideal (nonviscous and nonthermally conducting) gas with an adiabatic index $\varkappa$ which is described by the equations

$$\rho q^2\theta_L = -p_N, \quad \rho q^2\theta_N = -p_L\,(M^2 - 1) \tag{1.1}$$

where p and ρ are the pressure and density, q and θ are the modulus and the angle of inclination of the velocity vector, M is the Mach number and θ_L, p_L, θ_N and p_N are derivatives calculated along a flow line and along the normal to it.

As a consequence of (1.1), the expression for the derivative θ calculated along the isobars is [1]

$$\theta_l = -p_n\,(1 - M^2\sin^2\beta)/(\rho q^2) \tag{1.2}$$

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where p_n is the derivative calculated along a normal to the isobar and β is the angle between the isobar and the velocity vector.

When $M \leqslant 1$, the expression in parentheses is non-negative which makes relationship (1.2) useful in the study of subsonic flows.

Allowing for the possibility that there are branching points in subsonic flows [3], it will subsequently be understood that an isobar is a line of constant pressure which is always a boundary of a selected domain of increased or reduced pressure. In this case, the sign of p_n does not change along isobars and the equality $p_n = 0$ is only possible at the above-mentioned isolated branching points. By simply using the continuity of the pressure when $M \leqslant 1$, it can be shown that an isobar defined in this manner cannot break off within a subsonic flow domain. Actually, let the line $p = \text{const}$ be broken off at a certain internal point z or the sign of p_n change. An analysis of the pressure change along the neighbourhood of a sufficiently small radius with a centre at the point z then leads to a contradiction.

It follows from relationship (1.2) that the value of the angle θ changes monotonically along the isobars at each point of which $M \leqslant 1$ [1].

This fact is indicative of the following properties of subsonic flows which will subsequently be made use of.

Planar vortex flows cannot contain the following types of isobars at each point of which $M \leqslant 1$.

1. Closed isobars, excluding flows with closed stream lines or with internal stagnation points [1, 4].

2. Segments of isobars starting and finishing on a subsonic section of an SW [1]. The neighbourhood of a point is which the SW is perpendicular to the approach stream is also an exception subject to the condition that the SW at this point becomes convex towards the subsonic flow behind the SW [2].

3. Segments of isobars with terminal points in the straight wall, excluding flows with a stagnation point on the wall between these terminal points.

The assumption that there are no local supersonic zones in the domain under consideration plays an important role in investigating subsonic flows using isobars [1, 2]. As it turns out, the properties of subsonic flows noted above provide additional grounds for this assumption.

In vortex flows which are sufficiently close to being vortex-free when the entropy and the total enthalpy are close to constants, the isobars are sufficiently close to the lines $M = \text{const}$. It may therefore be asserted that there cannot be any local supersonic zones bounded by closed sonic lines or by the sonic lines in conjunction with a straight wall or SW in such flows.

Actually, in the opposite case isobars close to sonic lines exist at each point $M \leqslant 1$ which are closed or which reach the straight wall or the shock wave. However, this contradicts the properties of subsonic flows which have been considered above.

The following result is also concerned with the impossibility of local supersonic zones but it has no longer been assumed in its formulation that the flow is close to being vortex-free.

Theorem 1. A continuous planar vortex flow is considered in the neighbourhood of a non-convex wall *mn* (Fig. 1) along which the angle θ does not decrease. A subsonic flow runs from the left into the sonic line *ac* and a supersonic flow is realized on the right from *ac*. Each streamline intersects *ac* only once, that is, the stream function ψ does not decrease along *ac* and, finally, the total pressure $p_0(\psi)$ is a nondecaying function of ψ. (The latter condition is satisfied, for example, close to the wall of a symmetrical body around which there is a flow with the formation of a detached SW.)

It is asserted that, in the sonic line *ac*, it is impossible to choose a point *t* from which an isobar *ti*

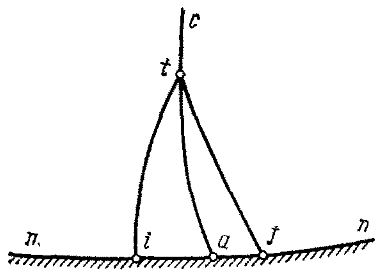

FIG. 1.

and a characteristic of a second family tj would emerge which reach subsonic and supersonic segments of the wall mn, respectively. (This fact is known in the case of vortex-free flow [5].)

Proof. Let us assume that such a point t exists. Then, the following compatibility condition is satisfied along the characteristic of the second family tj

$$d\theta - \frac{dp}{p} g(M, \varkappa) = 0, \quad g(M, \varkappa) = \frac{\sqrt{M^2-1}}{\varkappa M^2}$$

Allowing for the fact that $p = p_0(\psi)f(M, x)$, where $f(M, z)$ is an unknown function, we have

$$d\theta - \left(\frac{df}{f} + \frac{dp_0}{p_0}\right) g(M, \varkappa) = 0$$

Integrating, we get

$$\theta_j = \theta_t - h(M, \varkappa) - \int_0^{\psi_t} \frac{g(M, \varkappa)}{p_0(\psi)} p_0'(\psi)\, d\psi$$

$$h(M, \varkappa) = \sqrt{\frac{\varkappa+1}{\varkappa-1}} \operatorname{arctg} \sqrt{\frac{\varkappa-1}{\varkappa+1}(M^2-1)} - \operatorname{arctg} \sqrt{M^2-1}$$

By taking account of the fact that the derivative $p_0'(\psi) \geqslant 0$ according to the condition of the theorem and that $M > 1$, $h(M, x) > 0$, we find that $\theta_j < \theta_t$, that is, the angle θ is smaller at the point j than at the point t.

Let us now consider the isobar ti. It follows from the fact that the pressure does not decrease along ac that the normal derivative $p_n \leqslant 0$ on ti. Consequently, according to (1.2), $\theta_i > \theta_t$.

Finally, we have $\theta_j < \theta_t < \theta_i$. However, this contradicts the condition of the theorem, which it was required to prove.

Remark. As in Sec. 3 of this paper, proof of the fact that the isobar under investigation, which emerges from a point on the sonic line, actually reaches the segment of the wall being considered must precede the application of Theorem 1. We also note the inapplicability in Theorem 1 of the relationship describing the change in the angle θ along the sonic line [6]

$$\theta_s = \frac{q_\tau}{q} \cos^2 \varphi - \frac{p_s \operatorname{ctg} \varphi}{\rho q^2}$$

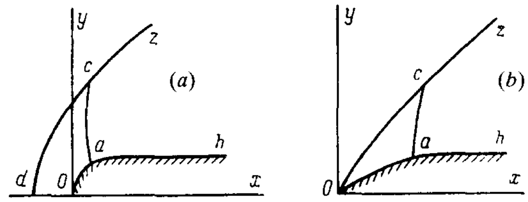

FIG. 2.

where θ_s, p_s and q_τ are derivatives calculated along the sonic line and along the normal to the sonic line and φ is the angle between the sonic line and the velocity vector.

In fact, it follows from the condition of the theorem that $\varphi \geqslant \pi/2$ in the neighbourhood of the wall *mn*. Hence, the terms on the right-hand side have different signs.

2. We now consider the upper half of the planar symmetric flow around a body of finite width *Oh* (Fig. 2) by a uniform, horizontal supersonic flow of an ideal (nonviscous and nonthermally conducting) gas with an adiabatic index $\varkappa$. Either a detached SW *dz* (Fig. 2a) or an attached SW *Oz* (Fig. 2b) is formed in front of the body. The subsequent treatment will be limited by the assumptions that subsonic flows without local supersonic zones and also zones with closed stream lines are realized in the domains *Odca* and *Oca*, where *ac* is the sonic line. The available data, including those obtained in Sec. 1, suggest that, in the case of a wide class of sufficiently smooth bodies, the above-mentioned assumptions are completely justified.

The flow diagram shown in Fig. 2(a) is realized in the case of blunt bodies and a wide class of pointed bodies. The flow in Fig. 2(b) is realized far less frequently. Certain results appertaining to this question are presented in [1, 2].

The values of p and θ on the shock waves *dz* and *Oz* are linked by the shock polar shown in Fig. 3. The polar is symmetric with respect to the $\theta = 0$ axis. $M = 1$ at the points c and c^- while above (below) these points $M < 1$ ($M > 1$). Finally, the entropy s decreases as the pressure p falls on the shock polar.

In studying the subsonic flows *Odca* and *Oca* (Fig. 2), one of the questions which arises is under what conditions do the points of just the right-hand half of the shock polar correspond to the segments *dc* and *Oc*? The following theorem is concerned with this question.

Theorem 2. Let the inclination of the wall be non-negative on the subsonic segment of the body, that is, $\theta \geqslant 0$ on *Oa*. The following then hold.

1. $\theta > 0$ on the subsonic sections of the shock waves *cd* and *Oc* and, as a consequence, the inclination of the shock wave does not exceed $\pi/2$ at each point of *dc* and *Oc*.

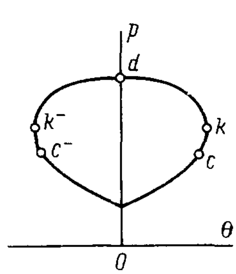

FIG. 3.

2. In the case of a detached shock wave, the pressure increases monotonically along an interval of the dO axis of symmetry.

Proof. Let us assume that the opposite is true. Let there be points on dc and Oc at which $\theta<0$. Then, by taking account of the continuity of p in subsonic flows and, as a consequence, the continuity of p and θ along dc and Oc, we can choose a point t on dc or Oc at which $\theta<0$ and there is an increase in θ and p as this point moves along dc or Oc. Consequently, the normal derivative $p_n \geq 0$ on the isobar which emerges from point t into the subsonic zone and the angle θ must decrease along it. Finally, the isobar cannot reach the body, the axis of symmetry or the shock wave. It also cannot reach the sonic line, since $p<p_c<p_t$ in the sonic line [2]. This inequality follows from the fact that $M \leq 1$ on dc and Oc according to the condition of the problem as a result of which the condition $p_0(\psi) \leq p_0(\psi_c)$ for the total pressure is completely valid on dc and Oc and, this means, also on the sonic line ac.

The resulting contradiction also proves the first part of the theorem.

Let us now assume that an acceleration of the flow is possible at certain points of the interval dO. We can then choose a point t at which $p>p_c$ and $p_x<0$. $p_n \geq 0$ on the isobar emerging from this point and, consequently, θ decreases along the isobar as a result of which the isobar cannot reach the shock wave, the axis of symmetry or the wall of the body. The isobar cannot reach the sonic line since $p>p_c$ on the isobar. The second part of the theorem is thereby proved.

Remark. Isobars, for which $p>p_c$, have been used in the proof of the theorem. It is therefore sufficient that the inequality $\theta \geq 0$ appearing in the condition of the theorem should only be satisfied to the left of the isobar ca^* joining point c to the wall. An analysis of the isobar ca^* shows that, at the point a^*, we have $M<1$, $\theta>\theta_c>0$. In particular, in the case of nonconcave bodies for which the angle θ cannot increase during a displacement to the right along the wall, to the left of the point a^* we have $\theta>\theta_c>0$. In other words, the theorem is automatically satisfied in the case of such bodies. We also note that, during the flow around nonconcave bodies, according to [1, 2], an attached shock wave with a subsonic flow behind it is only possible for a narrow range of values of θ_0, the angle of taper: $\theta_c<\theta_0<\theta_k$, where θ_k is the limiting angle of the shock polar.

The following theorem holds in the case of nonconcave bodies.

Theorem 3. During the flow around nonconcave bodies, the pressure p cannot increase along the subsonic segments dc and Oc of the shock wave and, as a consequence, the inclination of the shock waves also does not increase along these segments. In other words, the segments dc and Oc are nonconcave as are the bodies around which the flow occurs.

Proof. We recall that Theorem 2 holds in the case of the bodies being considered, that is, $\theta \geq 0$ on dc and Oc and the pressure does not decrease along dO. Let us now assume that an increase in the pressure is possible on dc and Oc. It follows from this and also from the assumption that $M<1$ on dc and Oc that there exists at least one point t where there is a local pressure maximum.

In the neighbourhood of this point, let us choose two points t^+ and t^- with different values of p and θ and with different signs of the derivatives p_n and θ_t on the isobars emerging from these points. On the isobar emerging from the point t^+ (it is located closer to the sonic line), $p_n \leq 0$ and $\theta_l \geq 0$ while, on the isobar emerging from the point t^-, we have $p_n \geq 0$, $\theta_l \leq 0$. Neither of these isobars can reach the shock wave in the interval dO of the axis of symmetry (a consequence of Theorem 2) nor can they reach the sonic line [2]. Consequently, they must reach the body at the points z^+ and z^-, respectively. We find from this that the angle θ is greater at the point z^+ than at the point z^-.

Furthermore, this is not excluded on a nonconcave wall since the point z^+ is located to the right of the point z^-. This contradiction proves the theorem.

As it turns out the nonconcavity of the subsonic segment of the shock wave can be proved when other conditions are imposed on the geometry of the body. The following theorem holds.

Theorem 4. If the condition $\theta \geqslant \theta_k$ is satisfied at each point of the body to the left of the isobar ca^* emerging from the sonic point of the shock wave, then the pressure does not increase along the subsonic section of the shock wave and, as a consequence, this section of the shock wave is concave.

The proof follows directly from an analysis of the isobars emerging from the points of the subsonic section of the shock wave at which, according to the assumption, the pressure increases. The point t^- from the proof of Theorem 3 may serve as an example.

In particular, bodies with nose cones and possibly also concave cones along which $\theta \geqslant \theta_k$ and which are joined through an angular point to the horizontal wall correspond to Theorem 4.

In the case of polytropic gases θ_k is only insignificantly greater than θ_c. It may therefore be assumed that the majority of bodies which satisfy Theorem 3 also satisfy Theorem 4 but the converse is not true.

Remark. The form of the shock arising during the flow around convex bodies has previously been studied in [7]. In the case of unbounded values of the Mach numbers, M_∞, of the approach stream, a conclusion was drawn concerning the convexity of the segment of the shock wave which is contiguous to the axis of symmetry on which $M < M^*$ $(M_\infty) < 1$. For example [7], when $x = 1.4$, the values $M^* \approx 0.97$, 0.88 and 0.60 correspond to the values of $M_\infty = 1.5$, 1.8 and 2.1, respectively. At the same time, all the results in the paper mentioned hold for any supersonic values of M_∞ and (Theorems 2–4) are satisfied along the whole of the subsonic section of the shock wave.

3. Let us now consider the circumfluence of a wedge Ogh of finite thickness with a break in the generatrix at a point g (Fig. 4). Here, a detached shock wave dz is shown which corresponds to a taper angle $\theta_0 > \theta_k$. However, the results of this section also refer to the narrow range of values of θ_0, $\theta_c \leqslant \theta_0 \leqslant \theta_k$ when an attached shock wave is realized which corresponds to a weak subsonic solution.

Thus, subsonic flow is realized in a certain domain between the wedge and the shock wave. The natural question arises regarding the position of the sonic line joining the body and the shock wave.

We know that the circumfluence of a convex angular point, analogous to point g in Fig. 4, is only possible with the formation of a supersonic flow with a local centred rarefaction wave in the neighbourhood of the point. Consequently, in the approach along the wall to the angular point, the

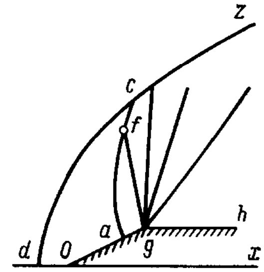

FIG. 4.

Mach number reaches a value $M_g \geqslant 1$. In other words, when $M_g = 1$, the sonic line reaches the angular point but, when $M_g > 1$, its starting point is located to the left.

In a number of problems involving vortex-free flows such as, for example, in the problem of the efflux of a jet from a vessel with rectilinear walls, it has been shown that the sonic line joins the terminal points of these walls [8–10]. These points are angular points for the boundary stream lines.

We also note that the reasoning regarding the position of the sonic line during the flow around a wedge with a break in the generatrix presented in [9] is only valid in the vortex-free approximation, which was also pointed out by the editors of the translation of [9]. Hence, the solution of the question of the position of the sonic line in the case of a vortex flow which occurs behind a curvilinear shock wave is of considerable interest. The following theorem is concerned with this question.

Theorem 5. Let us assume, as in Sec. 2, that the subsonic flow between the wall of the wedge, the shock wave and the closing sonic line does not contain any supersonic zones and domains with closed stream lines. We shall also assume that the stream function varies monotonically along the above-mentioned sonic line, that is, each of the stream lines can only intersect the sonic line once.

It is asserted that the sonic line which has a continuous supersonic flow adjacent to it on the right cannot start to the left of the break point of the contour.

Proof. Starting out from dimensionality considerations, it is necessary immediately to eliminate sonic lines from the treatment which do not fall within the domain of influence of a rarefaction fan with focusing of the characteristics belonging to the first family at the point g. Let us therefore consider a flow with a sonic line which starts to the left of point g, which the initial characteristic of the fan with its centre at point g, Fig. 4, reaches at point f.

The wedge is a nonconcave body. Hence, Theorems 2 and 3 are applicable to it and it follows from this that the pressure does not decrease along dO and does not increase along dc. The fact that there is no decrease in the total pressure $p_0(\psi)$ along dc also follows from the latter. By taking account of the hypothesis that there is no decrease in ψ along ac, we obtain that the pressure p also does not decrease along the sonic line ac.

It follows from the foregoing discussion that isobars which emerge from points of ac in the subsonic domain can only reach the interval Oa of the wall of the wedge. On the other hand, the characteristics of the second family which emerge from points of the interval ag reach the sonic line. Finally, by taking account of the fact that the wedge is also a nonconvex body, we find that all the conditions of Theorem 1 are satisfied, according to which the flow scheme being considered is impossible. Consequently, the theorem is proved.

So, during the circumfluence of a wedge with a break in the generatrix with a subsonic flow behind the shock wave in which the conditions of Theorem 5 are satisfied, the sonic line ac can only emerge from the angular point a (Fig. 5).

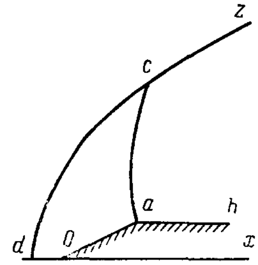

FIG. 5.

The monotonic change in the pressure along the wall of the wedge is proved in the following theorem.

Theorem 6. Let the conditions of the preceding theorem be satisfied. Then, the pressure does not increase along the wall of the wedge from the point of sharpening to the point of the break in the contour.

Proof. Let us assume that the opposite is true. Then, two points with different values of p but with different signs of the normal derivatives p_n on isobars emerging from these points can be chosen on the wall. The value of θ therefore increases along one of the isobars while decreasing along the other. Neither isobar can reach the straight wall Oa (Fig. 5). Meanwhile, a situation where both isobars reach different points of the contour $Odca$ is precluded.

The fact is that, as follows from the preceding theorems, the pressure does not increase on passing round the given contour in a clockwise direction. Consequently, there are not two points with the same pressure values on this contour. The resulting contradiction proves the theorem.

In concluding, we note that the results which have been obtained may also possibly not answer the question as to the existence of subsonic flows between a symmetric body and a shock wave which differ from those which have been considered above. However, these results do enable one to assert that, if such flows are possible, it only when the assumptions which have been made are not satisfied. In other words, either local subsonic zones must exist in the above-mentioned subsonic flows in this case, or domains with closed stream lines or which enclose sonic lines along which there is a nonmonotonic change in the stream function.

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